Dynamical invariants of categories associated to mapping tori

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Overview

Symplectic mapping torus

Let (M, ω) be a symplectic manifold and ϕ be a symplectomorphism. Define the symplectic mapping torus as

$$\bar{T}_\phi = M \times \mathbb{R} \times S^1 / (x, t, s) \sim (\phi(x), t+1, s)$$

It is a symplectic manifold fibered over T^2. Assume ϕ is not Hamiltonian.
Question: How can we distinguish $\bar{T}_\phi$ and $\bar{T}_{id_M} = M \times T^2$?

Answer: Assume M is compact and $H^1(M) = 0$. We can try to use an invariant called the Flux group to distinguish them.

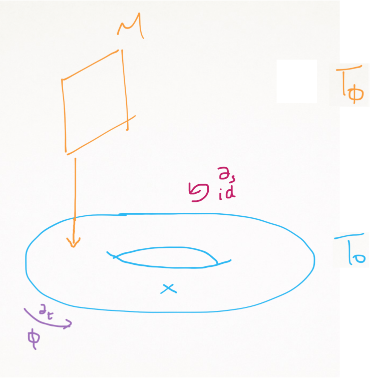
Flux

Given a compact symplectic manifold X, flux group is a discrete subgroup $\Gamma \subset H^1(X; \mathbb{R})$ which measures the aboundancy of loops/circles in the symplectomorphism group.

Applying this idea informally, $\bar{T}_{id_M} = M \times T^2$ admits circle actions in two independent directions(hence a rank 2-lattice many of them); whereas circle action in one direction is broken for $\bar{T}_\phi$.

This argument fails for

$$T_\phi = M \times (\mathbb{R} \times S^1 \setminus \mathbb{Z} \times 1)/(x, t, s) \sim (\phi(x), t + 1, s)$$



The circle action is broken on $T_0 = T^2 \setminus \{*\}$.

How to apply flux in this case?

- We may try to partially compactify T_ϕ
- Hard to characterize uniquely
- Heuristically partial compactifications correspond to deformations of the Fukaya category
- Hence, we wish to apply the idea of flux to $\mathcal{W}(T_\phi)$
- We propose an categorical model for the mapping torus and prove an abstract result instead

Advantage: Applies to manifolds X such that $\mathcal{W}(X) \simeq \mathcal{W}(T_\phi)$.

Work in progress: Have to relate the abstract categorical mapping tori to $\mathcal{W}(T_\phi)$.

Mapping torus categories

Let $\mathcal{A}$ be an A_∞ category over $\mathbb{C}$ and ϕ be an A_∞-autoequivalence. Further assume

1. $\mathcal{A}$ is smooth, i.e. the diagonal bimodule is perfect
2. $\mathcal{A}$ is proper in each degree and bounded below
3. $HH^i(\mathcal{A}) = 0$ for $i < 0$ and $HH^0(\mathcal{A}) \cong \mathbb{C}$

Associated to this data we construct a category M_ϕ, the **mapping torus category** satisfying the properties 1-3.

Sketch of the construction

Let $\tilde{\mathfrak{T}}_0$ denote the Tate curve. It is a chain of $\mathbb{P}^1$'s defined by gluing $Spec(\mathbb{C}[X_i, Y_{i+1}]/X_i Y_{i+1})$

Note the natural right translation automorphism $\mathfrak{tr} \curvearrowright \tilde{\mathfrak{T}}_0$ and the $\mathbb{G}_m$ action. Locally, $z \in \mathbb{G}_m$ acts by $X_i \mapsto z^{-1} X_i$, $Y_{i+1} \mapsto z Y_{i+1}$

We find a dg category $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ such that

1. $tw^\pi(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg})$ is a dg enhancement for $D^b(Coh_p(\tilde{\mathfrak{T}}_0))$, bounded derived category of coherent sheaves with a support of finite type
2. $\mathfrak{tr} = \mathfrak{tr}_*$ acts strictly on $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$
3. The geometric $\mathbb{G}_m$-action above induces a nice action on $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$

Moreover, $ob(\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}) = \{\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_i} : i \in \mathbb{Z}\}$.

Consider $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A}$, which carries a $\mathbb{Z}$-action generated by $\mathfrak{tr} \otimes \phi$.

Definition

The mapping torus category is defined as $M_\phi := (\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg} \otimes \mathcal{A})\#\mathbb{Z}$

Reminder on smash products

Given a dg category $\mathcal{B}$ with a (nice) action of the discrete group G, we can construct a category $\mathcal{B}\#G$ such that

1. $ob(\mathcal{B}\#G) = ob(\mathcal{B})$
2. $(\mathcal{B}\#G)(b, b') = \bigoplus_{g \in G} \mathcal{B}(g.b, b')$. Let $f \in \mathcal{B}(g.b, b')$ be denoted by $f \otimes g$
3. $(f \otimes g).(f' \otimes g') = fg(f') \otimes gg'$

Morally, if $\mathcal{B}$ has geometric origin this gives a category associated to quotient by G.

Remark

The $\mathbb{G}_m$-action on $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ induces a $\mathbb{G}_m$-action on M_ϕ.

Statement of the main theorem

We are now ready to state the main theorem:

Main theorem

Assume further $HH^1(\mathcal{A}) = HH^2(\mathcal{A}) = 0$*. If* M_ϕ *and* $M_{1_\mathcal{A}}$ *are Morita equivalent then* $\phi \simeq 1_\mathcal{A}$*.*

Reminder on Morita equivalences

Given two A_∞-categories $\mathcal{B}_1$ and $\mathcal{B}_2$, we call them Morita equivalent if there is a $\mathcal{B}_1$-$\mathcal{B}_2$-bimodule E and a $\mathcal{B}_2$-$\mathcal{B}_1$-bimodule E' such that $E \overset{L}{\otimes}_{\mathcal{B}_2} E' \simeq \mathcal{B}_1$ and $E' \overset{L}{\otimes}_{\mathcal{B}_1} E \simeq \mathcal{B}_2$. By Toen's work they are Morita equivalent if and only if $tw^\pi(\mathcal{B}_1)$ and $tw^\pi(\mathcal{B}_2)$ are A_∞-equivalent.

Algebro-geometric analogue

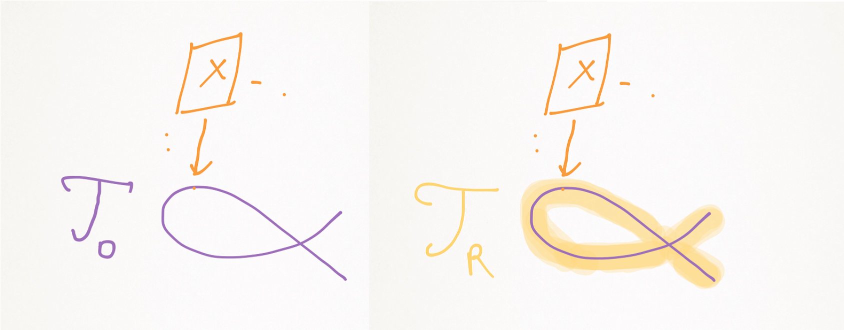
Given a variety X and automorphism $\phi_0 \curvearrowright X$ construct

$$M^{AG}_{\phi_0} = \tilde{\mathfrak{T}}_0 \times X/(t,x) \sim (\mathfrak{tr}(t), \phi_0(x)) \cong \mathbb{P}^1 \times X/(0,x) \sim (\infty, \phi_0(x))$$

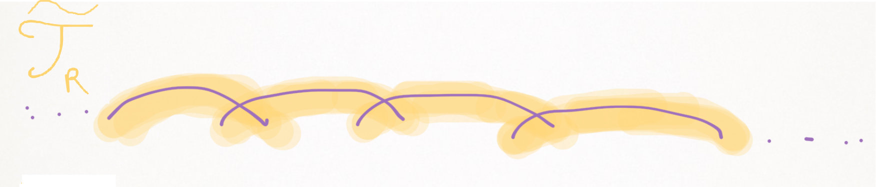
Remark

We expect $D^b(Coh(M^{AG}_{\phi_0})) \simeq H^0(tw^{\pi}(M_{\phi}))$ for $\phi = (\phi_0)_*$.

Before we sketch the proof of the main theorem let us give the basic idea on $M_{\phi_0}^{AG}$. $M_{\phi_0}^{AG}$ is fibered over $\mathcal{T}_0$, the nodal elliptic curve and it has a natural deformation over $Spf(R) = Spf(\mathbb{C}[[q]])$



Here $\mathfrak{T}_R$ denotes the Tate family, a natural smoothing of the nodal elliptic curve. One way to define the deformation $M_{\phi_0}^{AG,R}$ is to use the formal smoothing $\tilde{\mathfrak{T}}_R$ of $\tilde{\mathfrak{T}}_0$ locally given by $Spf(\mathbb{C}[X_i, Y_{i+1}][[q]]/(X_i Y_{i+1} - q))$



Then $M_{\phi_0}^{AG} := \tilde{\mathfrak{T}}_R \times X/(t,x) \sim (\mathfrak{tr}(t), \phi_0(x))$

Geometric idea

1. Pass to generic fiber $M_{\phi_0}^{AG,K}$ of $M_{\phi_0}^{AG,R}$ to obtain an analytic mapping torus over $K = \mathbb{C}((q))$
2. There is an action of the generic fiber $\mathcal{T}_K$ of $\mathcal{T}_R$ on $M_{1_X}^{AG,K} = \mathcal{T}_K \times X$(in a specific direction)
3. This action is broken on $M_{\phi_0}^{AG,K}$ unless $\phi_0 = 1_X$

Notice the same idea can be phrased in terms of $\mathbb{G}_{m,K}^{an}$-action on $M_{\phi_0}^{AG,K}$ which restricts to fiberwise action of ϕ_0 at $t = q$. This is essentially a flow line along a given direction. We will apply a categorical version of this idea, but instead of using generic fibers we will prove results up to q-torsion. Instead of flow lines, we will use family of "endo-functors" or bimodules parametrized by a formal scheme whose generic fiber gives $\mathbb{G}_{m,K}^{an}$, namely $\tilde{\mathfrak{T}}_R$.

- Need a categorical analogue of $M_{\phi_0}^{AG,R}$
- Deform $\mathcal{O}(\tilde{\mathfrak{T}}_0)_{dg}$ to obtain a curved dg category $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ over $R = \mathbb{C}[[q]]$ with action of $\mathfrak{tr}$
- Let $M_\phi^R := (\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A})\#\mathbb{Z}$

- We construct a family of endo-functors/bimodules of M_ϕ^R parametrized by $Spf(\mathbb{C}[u,t][[q]]/(ut-q)) \hookrightarrow \tilde{\mathfrak{T}}_R$
- First define it for $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ by utilizing a "graph" in $\mathcal{G}_R \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times Spf(A_R)$
- In local coordinates, $\mathcal{G}_R$ is given by

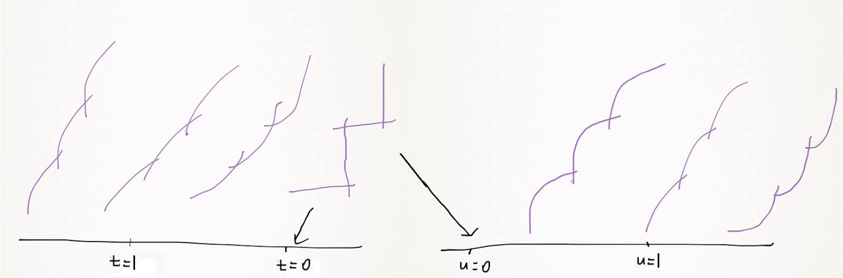
$$tY_{i+1} = Y'_{i+1}, tX'_i = X_i, Y_{i+1}X'_i = u \text{ or}$$

$$Y_{i+1} = uY'_i, X'_{i-1} = uX_i, Y'_iX_i = t$$

- This graph naturally extends to $\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R$ and in the generic fiber we expect the graph of $\mathbb{G}_{m,K}^{an} \times \mathbb{G}_{m,K}^{an} \to \mathbb{G}_{m,K}^{an}$ sending $(z_1, z_2) \mapsto z_1^{-1}z_2$

A picture of $\mathcal{G}_R|_{q=0}$

Imagine the part of $\mathcal{G}_R|_{q=0}$ on t-axis as degeneration of the action and the part on the u-axis as the degeneration of the inverse action composed with backwards translation.



$$\mathcal{G}_R|_{t=1} = \Delta_{\tilde{\mathfrak{T}}_R}, \mathcal{G}_R|_{u=1} = graph(\mathfrak{tr}^{-1})$$

The family of bimodules on M_ϕ^R

- First define an A_R-valued bimodule on $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$ by "$(\mathcal{F}, \mathcal{F}') \mapsto hom_{\tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R}(q^*\mathcal{F}, p^*\mathcal{F}' \otimes \mathcal{G}_R)$"
- Then descent to $M_\phi^R = (\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg} \otimes \mathcal{A})\#\mathbb{Z}$

We obtain an A_R-valued bimodule $\mathcal{G}_R$; hence, a module over $M_\phi^R \otimes (M_\phi^R)^{op} \otimes A_R$.

We prove $\mathcal{G}_R$ is a family of M_ϕ^R-bimodules(parametrized by $Spf(A_R)$) satisfying

1. $\mathcal{G}_R|_{q=0}$ can be represented by a twisted complex over $M_\phi \otimes M_\phi^{op} \otimes \mathbb{C}[u,t]/(ut)$.
2. The restriction $\mathcal{G}_R|_{t=1}$ is isomorphic to diagonal bimodule of M_ϕ^R
3. $\mathcal{G}_R$ follows the class $1 \otimes \gamma_\phi^R \in HH^1(M_\phi^R \otimes M_\phi^{R,op}, M_\phi^R \otimes M_\phi^{R,op})$ along the direction $t\partial_t - u\partial_u$

Here γ_ϕ^R is a distinguished class in $HH^1(M_\phi^R, M_\phi^R)$. We will explain the terms "family" and "follows". We show the properties 1-3 uniquely characterize the family $\mathcal{G}_R$ up to q-torsion.

Briefly families of (bi)modules

Given an A_∞-category $\mathcal{B}$ and a affine variety/formal scheme S, we can define **a family of (bi)modules parametrized by** S to be an (A_∞)-(bi)module $\mathfrak{M}$ over $\mathcal{B}$ which carries the structure of a (graded)free $\mathcal{O}(S)$-module such that the $\mathcal{B}$-(bi)module maps are $\mathcal{O}(S)$-linear. Define a morphism of families to be an A_∞ $\mathcal{B}$-(bi)module homomorphism that is $\mathcal{O}(S)$-linear.

We wish to measure the "rate of change" of the family along a derivation D_S on $\mathcal{O}(S)$.

For simplicity consider only families of right modules. Let $\mathfrak{M}$ be a family of right modules. Define **a pre-connection $\not{D}$ along D_S on** $\mathfrak{M}$ to be a collection of maps

$$\begin{aligned} \not{D}^1 :& \mathfrak{M}(b_0) \to \mathfrak{M}(b_0) \\ \not{D}^2 :& \mathfrak{M}(b_1) \otimes \mathcal{B}(b_0, b_1) \to \mathfrak{M}(b_0)[-1] \\ & \dots \end{aligned}$$

such that $\not{D}^i$ is $\mathcal{O}(S)$-linear for $i \geq 2$ and $\not{D}^1$ satisfies the Leibniz rule with respect to D_S, i.e. $\not{D}^1(fs) = f\not{D}^1(s) + D_S(f)s$.

$\not{D}$ can be thought as an A_∞-pre-module map and its differential, denoted by $def(\not{D})$ gives a class

$$def(\not{D}) \in hom^1_{\mathcal{B}^{mod}_{\mathcal{O}(S)}}(\mathfrak{M}, \mathfrak{M})$$

where $\mathcal{B}^{mod}_{\mathcal{O}(S)}$ is the category of families of right $\mathcal{B}$-modules parametrized by S. In particular, it is closed and $\mathcal{O}(S)$-linear and the cohomology class $[def(\not{D})]$ is independent of the choice of pre-connection $\not{D}$. Denote it by $Def(\mathfrak{M})$.

Let $\gamma \in CC^1(\mathcal{B},\mathcal{B})$. It induces an endomorphism of degree 1 on every $\mathcal{B}$-module and in particular a cochain

$$\gamma_{\mathfrak{M}}^{mod,0} \in hom^1_{\mathcal{B}^{mod}_{\mathcal{O}(S)}}(\mathfrak{M},\mathfrak{M})$$

If γ is closed and $[\gamma_{\mathfrak{M}}^{mod,0}] = Def(\mathfrak{M})$ we say $\mathfrak{M}$ follows γ.

Let $\mathcal{O}(S) = A_R := \mathbb{C}[u,t][[q]]/(ut-q)$ and $D_{A_R} := t\partial_t - u\partial_u$. This derivation can be seen as the infinitesimal action of $z\partial_z \in Lie(\mathbb{G}_m)$, where $z \in \mathbb{G}_m$ acts by $t \mapsto zt, u \mapsto z^{-1}u$.

Assume there is a (nice) $\mathbb{G}_m$-action on $\mathcal{B}$. Then again $z\partial_z \in Lie(\mathbb{G}_m)$ induces a class $(z\partial_z)^\# \in HH^1(\mathcal{B},\mathcal{B})$, the infinitesimal action.

Lemma

Assume a family $\mathfrak{M}$ carries a (nice) $\mathbb{G}_m$-equivariant structure. Then $\mathfrak{M}$ admits a natural pre-connection and follows the class $[(z\partial_z)^\#]$.

The graph $\mathcal{G}_R \subset \tilde{\mathfrak{T}}_R \times \tilde{\mathfrak{T}}_R \times Spf(A_R)$, which is locally given by

$$tY_{i+1} = Y'_{i+1}, tX'_i = X_i, Y_{i+1}X'_i = u \text{ or}$$

$$Y_{i+1} = uY'_i, X'_{i-1} = uX_i, Y'_iX_i = t$$

is $\mathbb{G}_m$-invariant, where $\mathbb{G}_m$ acts by $z : t \mapsto zt, u \mapsto z^{-1}u$ and $z : X'_i \mapsto z^{-1}X'_i$, $Y'_{i+1} \mapsto zY'_{i+1}$(i.e. trivially in the first component and as before in the second and third components).
Let $\gamma^R_\phi = (z\partial_z)^\#$:

Corollary

$\mathcal{G}_R$ *follows the class* $1 \otimes \gamma^R_\phi$.

Uniqueness of the family

Proposition

Let $\mathcal{G}'_R$ be another family of bimodules satisfying 1-3. Then, there exists morphisms $f : \mathcal{G}_R \to \mathcal{G}'_R$ and $g : \mathcal{G}'_R \to \mathcal{G}_R$ in the category $H^0((M^R_\phi)^{bimod}_{A_R})$- the homotopy category of families of bimodules- such that $f \circ g = q^N 1_{\mathcal{G}'_R}, g \circ f = q^N 1_{\mathcal{G}_R}$ for some N.

Hence, the family $\mathcal{G}_R$ is characterized by 1-3 up to q-torsion.

Proof of the uniqueness

Consider the chain complex $hom^{\cdot}_{(M^R_\phi)^{bimod}_{A_R}}(\mathcal{G}_R, \mathcal{G}'_R) = hom^{\cdot}(\mathcal{G}_R, \mathcal{G}'_R)$. It is a complex of flat A_R-modules and its cohomology is finitely generated over A_R in each degree(thanks to Property 1). This complex carries a connection along D_{A_R} in each degree given by

$$\text{“}\not{D}_{\mathcal{G}'_R} \circ (\cdot) - (\cdot) \circ \not{D}_{\mathcal{G}_R}\text{”}$$

Call such a collection of connections a pre-connection on the complex and denote it by $\not{D}$.

The class of $at(\not{D}) := d \circ \not{D} - \not{D} \circ d$ is given by

$$def(\not{D}_{\mathcal{G}'_R}) \circ (\cdot) - (\cdot) \circ def(\not{D}_{\mathcal{G}_R})$$

By Assumption 2 on families, $def(\not{D}_{\mathcal{G}_R})$, resp. $def(\not{D}_{\mathcal{G}'_R})$ is cohomologous to $\gamma^{mod,0}_{\mathcal{G}_R}$, resp. $\gamma^{mod,0}_{\mathcal{G}'_R}$(where $\gamma = 1 \otimes \gamma^R_\phi$); hence

$$at(\not{D}) \simeq \gamma^{mod,0}_{\mathcal{G}'_R} \circ (\cdot) - (\cdot) \circ \gamma^{mod,0}_{\mathcal{G}_R}$$

But this is null-homotopic, where the homotopy is given by a natural element $\gamma^{mod,1} : hom^0(\mathcal{G}_R, \mathcal{G}'_R) \to hom^0(\mathcal{G}_R, \mathcal{G}'_R)$.

Let C^* be a complex of A_R-modules and endow each C^i with a connection along D_{A_R}. Let $\not{D}$ denote this pre-connection. As before,
$at(\not{D}) := d(\not{D}) = d \circ \not{D} - \not{D} \circ d$.

Lemma

Assume $at(\not{D}) = d(h) = d \circ h - h \circ d$ for $h \in hom^0(C^*, C^*)$. Then, h can be used to correct $\not{D}$ so that $\not{D}$ becomes a chain map.

In particular, $hom^\cdot(\mathcal{G}_R, \mathcal{G}'_R)$ is a complex of A_R-modules with connections and the collection of connections form a chain map.

Corollary

$Hom(\mathcal{G}_R, \mathcal{G}'_R) = H^0(hom^\cdot(\mathcal{G}_R, \mathcal{G}'_R))$ is a finitely generated A_R-module with a connection.

Remark

The special choice $\gamma^{mod,1}$ of null-homotopy makes sure that compositions such as

$$Hom(\mathcal{G}'_R, \mathcal{G}_R) \otimes_{A_R} Hom(\mathcal{G}_R, \mathcal{G}'_R) \to Hom(\mathcal{G}_R, \mathcal{G}_R)$$

are also compatible with the induced connections.

Before proceeding the proof of uniqueness, let us make a remark about $Hom(\mathcal{G}_R, \mathcal{G}'_R)|_{t=1}$. As expected, it is isomorphic to $Hom(\mathcal{G}_R|_{t=1}, \mathcal{G}'_R|_{t=1})$ but this relies on the existence of connection on the complex $hom(\mathcal{G}_R, \mathcal{G}'_R)$.

Lemma

$HH^0(M^R_\phi, M^R_\phi) \cong R.$

Corollary

$Hom(\mathcal{G}_R, \mathcal{G}'_R)|_{t=1} = Hom(\mathcal{G}_R, \mathcal{G}'_R)/(t-1)Hom(\mathcal{G}_R, \mathcal{G}'_R) \cong R.$

The rest of the proof of uniqueness depends on commutative algebra of modules over $A_R = \mathbb{C}[u,t][[q]]/(ut-q)$.

Lemma

Let M be a finitely generated A_R-module which carries a connection D_M along D_{A_R}. Assume $M|_{t=1} = M/(t-1)M$ is q-torsion over $A_R/(t-1)A_R = R$. Then M is q-torsion.

Lemma

Let M be a finitely generated A_R-module which carries a connection D_M along D_{A_R}. Then, M is free up to q-torsion.

Corollary

$Hom(\mathcal{G}_R, \mathcal{G}'_R)$ is isomorphic to $A_R = \mathbb{C}[u,t][[q]]/(ut-q)$ up to q-torsion.

Consider

$$Hom(\mathcal{G}'_R, \mathcal{G}_R) \otimes_{A_R} Hom(\mathcal{G}_R, \mathcal{G}'_R) \to Hom(\mathcal{G}_R, \mathcal{G}_R)$$

All the modules involved carry connections and $Hom(\mathcal{G}'_R, \mathcal{G}_R)$ etc. are isomorphic to A_R up to q-torsion. Up to q-torsion it is equivalent to

$$A_R \otimes_{A_R} A_R \to A_R$$

and thus we can lift $q^N 1_{\mathcal{G}_R}$ for some N. Same in the other direction.

Proof of the main theorem

Assume M_ϕ and $M_{1_\mathcal{A}}$ are Morita equivalent.

Claim

M_ϕ^R is Morita equivalent to $\psi_q^* M_{1_\mathcal{A}}^R$ where ψ_q is a transformation of R.

This holds since the only deformation of M_ϕ that is non-trivial in the first order is M_ϕ^R. For simplicity assume $\psi_q = 1_R$ and M_ϕ^R is Morita equivalent to $M_{1_\mathcal{A}}^R$.

Claim

$HH^1(M_\phi^R, M_\phi^R) \cong HH^1(M_{1_\mathcal{A}}^R, M_{1_\mathcal{A}}^R) \cong R^2$ and the Morita equivalence can be modified so that the natural isomorphism carries γ_ϕ^R to $\gamma_{1_\mathcal{A}}^R$.

$M_{1_\mathcal{A}}^R \simeq \mathcal{A} \otimes M_{1_\mathbb{C}}^R$ and $M_{1_\mathbb{C}}$ is a model for $D^b Coh(\mathfrak{T}_0) \simeq D^\pi \mathcal{W}(T_0)$ where $\mathfrak{T}_0$ is the nodal elliptic curve and T_0 is the punctured torus. Hence, it has sufficient symmetries to modify the Morita equivalence.

Remark

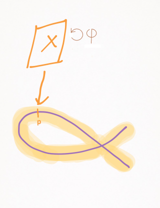
Heuristically, $HH^1(\mathcal{B}, \mathcal{B})$ can be thought as the Lie algebra of $Auteq(tw^\pi(\mathcal{B}))$. In our situation we have a natural copy of $\mathbb{Z}^2$ inside $HH^1(\mathcal{B}, \mathcal{B})$- the coroots- and the classes above fall into these discrete subgroups.

The Morita equivalence gives a correspondence between families of bimodules parametrized by $Spf(A_R)$. Moreover, the family $(\mathcal{G}_R)_{1_{\mathcal{A}}}$ corresponds to still satisfies 1-3. Hence, it is the same as $(\mathcal{G}_R)_\phi$ up to q-torsion.

Remark

$(\mathcal{G}_R)_{1_{\mathcal{A}}}|_{u=1}$ is isomorphic to diagonal and $(\mathcal{G}_R)_\phi|_{u=1}$ is isomorphic to "fiberwise ϕ".

Fiberwise ϕ is an auto-equivalence of M_ϕ^R that is given by the descent of $\mathfrak{tr}^{-1} \otimes 1_{\mathcal{A}}$ or $1_{\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}} \otimes \phi$ on $\mathcal{O}(\tilde{\mathfrak{T}}_R)_{cdg}$. This implies fiberwise ϕ is the same as $1_{\mathcal{A}}$ up to q-torsion.



Pick a smooth R-point p on the deformation of nodal curve. Any $a \in ob(\mathcal{A})$ we have an unobstructed object "$\mathcal{O}_p \otimes a$" over M_ϕ^R and a subcategory $\{\mathcal{O}_p\} \otimes \mathcal{A}$. Fiberwise ϕ induces $1 \otimes \phi$ on $\{\mathcal{O}_p\} \otimes \mathcal{A}$ and it is the same as the diagonal bimodule up to q-torsion. Hence, after inverting q, they are the same and this easily implies $\phi \simeq 1_{\mathcal{A}}$.